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LETTER TO THE EDITOR

Exact solubility of the self-dual and the string melting points in the restricted solid-on-solid model

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Abstract. The restricted solid-on-solid model, which describes the roughening of two-dimensional interfaces, is found to be exactly soluble in two instances: at a point where the model is self-dual under the duality transformation of the five-state clock model and at the string melting transition where the model belongs to the same universality class as an Ising model on a transverse vibrating lattice. Both points are located in the subspace of the 19-vertex model solved by Zamolodchikov and Fateev.

Interconnections between various models in two-dimensional statistical mechanics have proven to be a valuable source of information for determining their critical behaviour. It often happens that self-dual points and critical points of different models can be mapped onto each other, e.g. the self-dual lines in the Potts, Ashkin-Teller and eight-vertex model (see, e.g., den Nijs 1979) and moreover that at these points the models are exactly soluble. One of us pointed out recently that the restricted solid-on-solid (RSOS) model (den Nijs 1985b) (which is not the model of Andrews *et al* (1984)), with only nearest-neighbour interactions, is self-dual at one specific temperature under the duality transformation of the five-state clock model (den Nijs 1985a, b, c) which contains the RSOS model as a limiting case (Domany *et al* 1980). Another special point is located at antiferromagnetic coupling. There the model undergoes a string melting transition which belongs to the same universality class as that in an Ising model on a transverse vibrating lattice (den Nijs 1985b). Guided by past experience, we suspect that the RSOS model might also be exactly soluble at these points. In this letter we show that this is indeed the case: both points are in fact special cases of a 19-vertex model solved recently by Zamolodchikov and Fateev (1980).

Solid-on-solid models describe roughening transitions in two-dimensional interfaces (for a review see Weeks 1980). The interface is characterised with respect to a reference plane by integer-valued column height variables h_r, at sites r of the lattice. In the RSOS model the step height between nearest-neighbour sites is restricted to $dh(r, r') = 0, \pm 1$. The most general Hamiltonian, with interactions between all four columns in each elementary plaquette on a square lattice (see figure 1) has four independent

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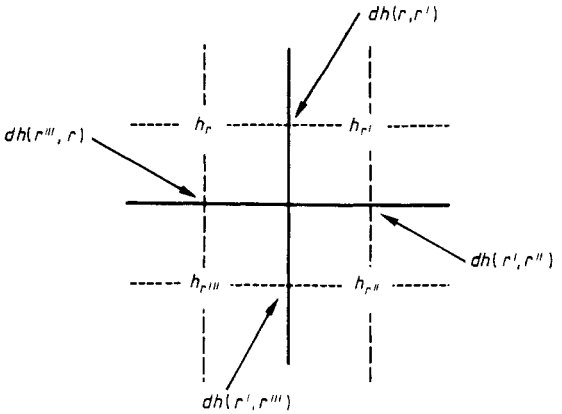

Figure 1. The column height variables h_r and step variables $dh(r, r')$ at an elementary plaquette. The broken lines represent the square lattice of the RSOS model and the full lines the lattice of the corresponding 19-vertex model.

coupling constants if we assume step-up step-down symmetry and that the interactions are isotropic, i.e.

$$\mathcal{H} = \sum_{\langle r,r' \rangle} J\delta(h_r - h_{r'}, 0) + \sum_{(r,r'')} \{L_1\delta(h_r - h_{r''}, \pm 1) + L_2\delta(h_r - h_{r''}, \pm 2)\} + \sum_{\{r,r',r'',r'''\}} Q\delta(h_r - h_{r''}, \pm 1)\delta(h_{r'} - h_{r'''}, \pm 1) \tag{1}$$

where $\langle\ ,\ \rangle$ denotes summation over nearest neighbours, $(\ ,\ )$ over next-nearest neighbours, and $\{\ ,\ldots,\ \}$ over the elementary plaquettes. As usual, $\mathcal{H}$ and the coupling constants are dimensionless quantities, divided by $(-k_B T)$.

For the sake of clarity, it is useful to recall the behaviour of the RSOS model along the nearest-neighbour axis, $L_1 = L_2 = Q = 0$ (den Nijs 1985a, b, c). There are three phases in the interval going from antiferromagnetic to ferromagnetic zero temperature, i.e. from $J = -\infty$ to $J = +\infty$: the BCSOS rough phase, the RSOS rough phase and the ferromagnetic flat-ordered phase. This ordered phase has a finite correlation length ξ, but in both rough phases, $\xi = \infty$. In the latter, the roughness of the interface can be characterised by the amplitude of the logarithmic divergence of the height–height correlation function

$$\langle (h_{r_0+r} - h_{r_0})^2 \rangle \simeq \frac{1}{\pi K} \ln(|r|). \tag{2}$$

At $J = -\infty$ the model reduces to the exactly soluble body-centred solid-on-solid (BCSOS) model (van Beijeren 1977), where a step must be present at each bond, i.e. $dh(r, r') = \pm 1$. The 'zeros' $dh(r, r') = 0$ are frozen out. The BCSOS model is at the so-called ice point, which implies that the interface remains rough with $K = \pi/6$. At finite $J < J_s$ neighbouring equal heights ($dh(r, r') = 0$) are allowed but do not contribute to the fractal critical fluctuations of the step structure yet. These 'zeros' form strings of closed loops with a typical diameter ζ_0 inversely proportional to their string tension. Thus the rough phase contains loop-like impurities of size ζ_0, but remains of BCSOS type.

The model belongs to the same universality class as an Ising model on a transverse vibrating lattice because the loops of bonds with $dh(r, r') = 0$ can be identified with Bloch walls of an Ising model and because the $dh(r, r') = \pm 1$ BCSOS steps of the rough interface can be identified with lattice vibrations. The 'zeros', i.e. the Ising Bloch walls, reduce the height fluctuations. It is as their presence stiffens the elastic constants locally (den Nijs 1985b).

At $J_s = -0.4815$ the Ising spins disorder: the loops of 'zeros' melt, due to meander entropy. That this transition is indeed of Ising type has been confirmed numerically with finite-size scaling methods (den Nijs 1985b). At $J = J_s$ the 'zeros' start to participate in the fractal step structure. For $J_s < J < J_R$ the model is in the RSOS rough phase.

At increasing J, the 'zeros' reduce the roughness K further, until $J_R = 0.633$; K then becomes equal to the universal value $K = \pi/2$ where a Kosterlitz-Thouless (KT) transition takes place into the flat-ordered phase. The KT nature of this roughening transition has also been confirmed numerically (see, e.g., Luck 1981, den Nijs 1985c).

Inside the RSOS rough phase there is one special point $J_{SD} = 1/2(1+\sqrt{5}) = 0.4812$ at which the model is self-dual under the duality transformation of the five-state clock model. The self-duality implies that $K = 2\pi/5$ and also that corrections to scaling have a special behaviour at $J = J_{SD}$ (den Nijs 1985c).

Zamolodchikov and Fateev (1980) showed that the RSOS model, defined by equation (1), is exactly soluble in a one-dimensional subspace. Of course, they did not originally present their result in the RSOS model language. They constructed a special class of spin-1 factorisable, elastic, unitary, *CPT* and charge symmetry invariant two-dimensional S matrices and showed that their solution allows one to construct a commuting family of transfer matrices for a 19-vertex model. This 19-vertex model is equivalent to the RSOS model, as pointed out by Sogo *et al* (1983a, b) if one identifies the step states $dh(r, r') = -dh(r', r) = h_r - h_{r'}$, $=0, \pm 1$ on the bonds of the two-dimensional square lattice of figure 1 with the spin-1 variables. The so-called 'ice rule' in the vertex model (Lieb 1967), which states that all contour integrals round a site over the step variables are zero (the absence of screw dislocations in the interface) is equivalent to conservation of spin momentum.

Zamolodchikov and Fateev solved the model in a two-dimensional subspace described by two parameters α and μ. The first parameter corresponds to a lattice anisotropy in the next-nearest-neighbour interactions and takes the values $\pi/2$ in the isotropic case. The exactly soluble line in the four-dimensional subspace of equation (1) is now parametrised by μ as

$$\begin{aligned} \sigma &= \exp(2J) = 1 - t \\ t &= \pm\exp(Q + J + 2L_1) = \sin^2 \mu (\sin 2\mu \sin 4\mu)^{-1} \\ a &= \exp(J + L_1) = \sin \mu (\sin 2\mu)^{-1} \\ s &= \exp L_2 = \sin \mu \sin 3\mu (\sin 2\mu \sin 4\mu)^{-1} \end{aligned} \tag{3}$$

where σ, t, a, s label the Boltzmann weights in the 19-vertex model used by Zamolodchikov and Fateev. μ is real along one part of this solubility line and switches to pure imaginary values at $\mu = 0$. The projection of this line in the (J, L_2) plane is shown in figure 2. Note that there are two separate branches at positive Boltzmann weights.

The main point of this letter is the observation that the solubility line, equation (3), intersects the nearest-neighbour axis (e.g. $L_1 = L_2 = Q = 0$) at precisely two points, at $\mu = \pi/5$ and $2\pi/5$. These points are special points of the RSOS model. For $\mu = 2\pi/5$

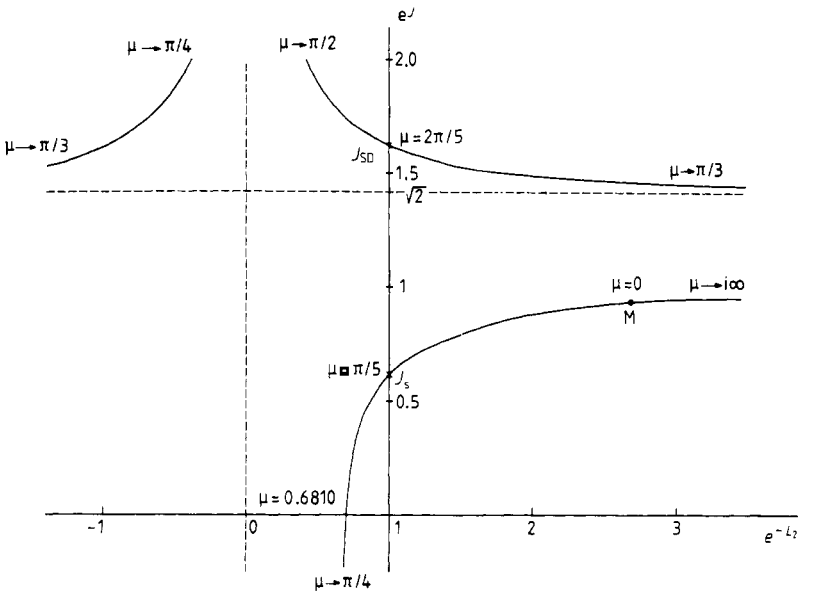

Figure 2. Projection of the exactly soluble line, equation (3), on the (J, L_2) plane.

we find $e^J = 1/2(1+\sqrt{5})$ which is the self-dual point $J = J_{SD}$ (den Nijs 1985b). For $\mu = \pi/5$, equation (3) yields $e^J = 1/2(\sqrt{5}-1)$ and the corresponding value of J is so close to the numerical estimate of the string melting coupling $J_s = -0.4815 \pm 0.0005$ (den Nijs 1985b) that we can identify it with J_s. Also notice that at J_{SD}, μ happens to be equal to the value of the roughness parameter $K = 2\pi/5$. At J_s the value of K has been estimated numerically: $K = 0.638 \pm 0.006$ (den Nijs 1985b). This value is close enough to $\pi/5$ to conclude that also at J_s, $\mu = K = \pi/5$.

One interesting prospect is to change the parameters of the model such that the string melting at J_s and the roughening at J_R approach each other and start to interfere. This takes place in the generalised Hamiltonian (see equation (1)). A systematic discussion and numerical study using finite-size scaling methods is in progress (Rommelse and den Nijs 1986) for the rich phase diagram of the generalised RSOS model. Here we simply point out how the exact soluble line fits into it.

Sogo *et al* (1983a, b) calculated the free energy along the exactly soluble line. They report first-order KDP-like singularities in the free energy at $\mu = \pi/2$, $\pi/4$. At these values of μ the exactly soluble line crosses over to another branch. They also report a roughening transition with a Kosterlitz-Thouless-type infinite-order singularity in the specific heat at $\mu = 0$, where μ crosses over from real to pure imaginary values (point M in figure 2). They further suggest that, although the interactions along the exactly soluble line are somewhat complicated, the model behaves essentially the same as the BCSOS model. However, a close inspection of their free energy expressions in the neighbourhood of $\mu = 0$ reveals no such infinite-order singularity.

The free energy for μ real and $v = (2\alpha\mu/\pi)$ arbitrary is given by equation (2.23) of Sogo *et al* (1983b):

$$f_+(\alpha, \mu) = \ln \rho_0 + \sum_{p=1}^{2S} \int_{-\infty}^{\infty} \cosh(\pi - 4\overline{p\mu})t \frac{\sinh vt \sinh(2\mu - v)t}{t \sinh \pi t \cosh 2\mu t} \mathrm{d}t. \tag{4}$$

This expression also applies to a generalisation of the spin-1 model to a $q=(2S+1)$-state vertex model (Sogo *et al* 1983a, b). ρ_0 is a normalisation constant. Our μ parameter is one half of that used by Sogo *et al*. The symbol $\overline{p\mu}$ means the excess of μ over a given n, $\overline{p\mu}=p\mu-n\pi$, so for μ small $\overline{p\mu}=p\mu$. Using the following identity:

$$\sum_{p=1}^{2S}\cosh(\pi-4p\mu)t=2\cosh 2\mu t\sum_{p=1}^{S}\cosh[\pi-(2p+1)2\mu]t \tag{5}$$

which applies to odd values of q only, the integrand in equation (4) simplifies considerably and in fact the integration can be explicitly done with the formula (Oberhettinger 1957)

$$\int_0^\infty\frac{\cosh ax-\cosh bx}{x\sinh cx}\cos xy\,dx=\tfrac{1}{2}\ln\frac{\cos(\pi b/c)+\cosh(\pi y/c)}{\cos(\pi a/c)+\cosh(\pi y/c)}. \tag{6}$$

The result is then (always for small μ)

$$f_+(\alpha,\mu)=\ln\rho_0+\sum_{n=1}^{S}\ln\frac{\sin(2p\mu+v)\sin[(p+1)2\mu-v]}{\sin 2p\mu\sin 2(p+1)\mu}. \tag{7}$$

Now at purely imaginary $2i\mu=\lambda$ and $iv=u$, the free energy is represented by the infinite series; see equation (2.14) in Sogo *et al* (1983b):

$$f_-(u,\lambda)=\ln\rho_0+2\sum_{n=1}^{\infty}\left(\sum_{p=1}^{2S}\exp(-2\lambda np)\right)\frac{\sinh nu\sinh(\lambda-n)n}{n\cosh n\lambda}. \tag{8}$$

Again the identity, for s integer (q odd),

$$\sum_{p=1}^{2S}\exp(-2\lambda np)=2\cosh n\lambda+\sum_{p=1}^{S}\exp[-\lambda n(2p+1)] \tag{9}$$

allows us to simplify the series of equation (8) and the summation may be explicitly performed first over p, then over n, by using formula (1.78) of Oberhettinger (1973). We obtain

$$f_-(u,\lambda)=\ln\rho_0\prod_{p=1}^{S}\frac{\sinh(p\lambda+u)\sinh[(p+1)\lambda-u]}{\sinh p\lambda\sinh(p+1)\lambda}. \tag{10}$$

Thus the transition from equation (7) to equation (10) is analytic including our isotropic case where $v=\mu$ ($\mu=\lambda/2$). This result agrees with that of Zamolodchikov and Fateev (1980). Thus an essential singularity can only exist for $q=$ even, in particular for $q=2$ (van Beijeren 1977).

Point M is probably a high-order multicritical point in the phase diagram. Not only point J_s on the nearest-neighbour axis, but the entire branch of the exactly soluble line with $0<\mu<\pi/4$ must be part of the string melting critical surface. Otherwise our Ising singularity should have been observed along this as at J_s. Moreover, the part of this line beyond M where μ is pure imaginary is likely to be part of the coexistence surface between the ferro- and antiferromagnetic flat-ordered interface phases. In the limit $i\mu\to\infty$, it lies in the plane $J=0$ which is the asymptotic equation for this coexistence plane as $L_2\to-\infty$. So M is likely to be the multicritical point in the phase diagram where the roughening transition, string melting transition and coexistence surfaces meet. Along the exactly soluble line this point is approached in a special way such that no singularity in the free energy is seen.

Note that the roughness parameter K cannot be equal to μ along the entire exactly soluble line. At J_S and J_{SD}, $\mu = K$, but at the roughening point M where $\mu = 0$, K should take the universal value $\pi/2$.

The other branch of the exactly soluble line, which goes through the self-dual point J_{SD} at $\mu = 2\pi/5$, must lie entirely inside the RSOS rough phase. There are no singularities in the free energy nor special points where the nature of the solution changes. It is simple to convince oneself that in the limit $\mu = \pi/3$, where $L_2 \to \infty$ while $J = -L_1 = Q = \frac{1}{2}\ln 2$, this line indeed remains inside the RSOS rough phase, using simple estimates for the meander contributions to the step free energies (Rommelse and den Nijs 1986). This branch of the exactly soluble line does not seem to be special, except that, as at the self-dual point J_{SD}, the corrections to scaling probably simplify.

References

Andrews G E, Baxter R J and Forrester P J 1984 *J. Stat. Phys.* **35** 193

den Nijs M 1979 *J. Phys. A: Math. Gen.* **12** 1857

—— 1985a *Phys. Rev.* B **31** 266

—— 1985b *J. Phys. A: Math. Gen.* **18** L549

—— 1985c *Phys. Rev.* B **32** 4785

Domany E, Mukamel D and Schwimmer A 1980 *J. Phys. A: Math. Gen.* **13** L311

Lieb E H 1967 *Phys. Rev. Lett.* **18** 1046

Luck J M 1981 *J. Physique Lett.* **42** 275

Oberhettinger F 1957 *Tabellen zur Fourier Transformation* (Berlin: Springer) p 37

—— 1973 *Fourier Expansions* (New York: Academic) p 15

Rommelse J and den Nijs M 1986 unpublished

Sogo K, Akutsu Y and Abe T 1983a *Prog. Theor. Phys.* **70** 730

—— 1983b *Prog. Theor. Phys.* **70** 739

van Beijeren H 1977 *Phys. Rev. Lett.* **38** 993

Weeks J D 1980 *Ordering of Strongly Fluctuating Condensed Matter Systems* ed T Riste (New York: Plenum) p 293

Zamolodchikov A B and Fateev V A 1980 *Sov. Nucl. Phys.* **32** 298